NEW HIGH ORDER SUFFICIENT CONDITIONS FOR CONFIGURATION TRACKING

M. BARBERO-LIÑÁN, M. SIGALOTTI

ABSTRACT. In this paper, we propose new conditions guaranteeing that the trajectories of a mechanical control system can track any curve on the configuration manifold. We focus on systems that can be represented as forced affine connection control systems and we generalize the sufficient conditions for tracking known in the literature. The new results are proved by a combination of averaging procedures by highly oscillating controls with the notion of kinematic reduction.

1. INTRODUCTION

New geometric techniques are used to generalize tracking conditions known in the literature [1, 2, 3]. The tracking problem plays a key role in the performance of robots and mechanical systems such as submarines and hovercrafts in order to avoid obstacles, stay nearby a preplanned trajectory, etc.

Mechanical control systems are control-affine systems on the tangent bundle of the configuration manifold Q. In order to simplify the motion planning tasks for these control systems, a useful tool has been introduced in the geometric control literature, namely, the notion of kinematic reduction. Such a procedure consists in identifying a control-linear system on Q whose trajectory mimic those of the mechanical system. This approach has been useful to describe controllability, planning properties [1] and optimality [4] of mechanical systems. However, as described in [1], kinematic reduction is not always possible, some conditions related to the symmetric closure of the control vector fields of both systems under study must be satisfied.

In our previous work [2] we extended the first-order sufficient conditions for tracking proposed in [1] by using different families of vector fields, possibly of infinite cardinality. Related constructions to generate admissible directions for tracking have been proposed in [5, 6] (see also [7, 8]).

Our first goal in this current paper is to establish a relationship between families of vector fields defined pointwise and sets of sections of the tangent bundle defined in a recurrent way, similarly to the classical Malgrange theorem [9]. This new pointwise characterization of families of vector fields used in [2] allows to use kinematic reduction in order to obtain more general sufficient tracking conditions. As a result, it can be proved that an underwater vehicle with a natural choice of control vector fields is always trackable, even in the most symmetric case (see the example in Section 4.4 for more details).

The paper is organized as follows. Section 2 contains all the necessary background in forced affine connection control systems [1]. Section 3 defines the notion of trackability under study. After recalling in Section 4.1 the high-order tracking conditions known in the literature and obtained by averaging theory [2], kinematic reduction is used to obtain more general tracking conditions in Section 4.3. The full characterization of the trackability of the system describing the motion of an underwater vehicle is achieved in Section 4.4, concluding the study started in [3] and continued in [2].

2. Notation and preliminaries

Denote by $\mathbb{N}$ the set of positive natural numbers and write $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. Fix $n \in \mathbb{N}$. From now on, Q is a n–dimensional smooth manifold and $\mathfrak{X}(Q)$ denotes the set of smooth vector fields on Q. All vector fields are considered smooth as functions on Q, unless otherwise stated. Let $\tau_Q \colon TQ \to Q$ be the canonical tangent projection. A *vector field along* τ_Q is a mapping $X \colon TQ \to TQ$ such that $\tau_Q \circ X = \tau_Q$. We denote by I a compact interval of the type $[0, \tau]$, $\tau > 0$.

2.1. **Affine connection control systems.**
The trajectories $\gamma : I \to Q$ of a Lagrangian mechanical systems on a manifold Q are minimizers of the action functional

$$A_L(\gamma) = \int_I L(t, \dot{\gamma}(t)) \mathrm{d}t$$

associated with a Lagrangian function $L \colon \mathbb{R} \times TQ \to \mathbb{R}$.

The solutions to this variational problem must satisfy the well-known Euler–Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n, \tag{1}$$

where (q^i, v^i) are local coordinates for TQ. Here we consider controlled Euler–Lagrange equations obtained by modifying the right-hand side on the above equation, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = \sum_{a=1}^{k} u_a Y_a^i, \quad i = 1, \dots, n,$$

with $u_a \colon I \to \mathbb{R}$, $Y_a^i \colon Q \to \mathbb{R}$.

When the manifold Q is endowed with the Riemannian structure given by a Riemannian metric g and the Lagrangian function $L_g(v_q) = \frac{1}{2} g(v_q, v_q)$ is considered, the solutions to (1) turn out to be the geodesics of the Levi–Civita affine connection ∇^g associated with the Riemannian metric. (See [1] for more details and for many examples of mechanical control systems that fit in this description.)

When control forces are added to the geodesic equations we obtain an affine connection control system

$$\nabla^g_{\dot{\gamma}(t)} \dot{\gamma}(t) = \sum_{a=1}^{k} u_a(t) Y_a(\gamma(t)),$$

where $Y_1, \dots, Y_k$ are vector fields on Q.

The notion of affine connection control system can be extended without the need of the Levi–Civita connection.

Definition 2.1. *An **affine connection** is a mapping*

$$\begin{aligned} \nabla \colon \mathfrak{X}(Q) \times \mathfrak{X}(Q) &\longrightarrow \mathfrak{X}(Q) \\ (X, Y) &\longmapsto \nabla(X, Y) = \nabla_X Y, \end{aligned}$$

satisfying the following properties:

(1) ∇ *is $\mathbb{R}$–linear on X and on Y;*
(2) $\nabla_{fX} Y = f \nabla_X Y$ *for every $f \in \mathcal{C}^\infty(Q)$;*
(3) $\nabla_X fY = f \nabla_X Y + (Xf) Y$, *for every $f \in \mathcal{C}^\infty(Q)$. (Here Xf denotes the derivative of f in the direction X.)*

The mapping $\nabla_X Y$ is called the *covariant derivative of Y with respect to X*. Given local coordinates (q^i) on Q, the *Christoffel symbols for the affine connection* in these coordinates are given by

$$\nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^r} = \sum_{i=1}^{n} \Gamma^i_{jr} \frac{\partial}{\partial q^i}, \qquad j, r = 1, \dots, n.$$

From the properties of the affine connection, we have

$$\nabla_X Y = \sum_{i,j,r=1}^{n} \left(X^j \frac{\partial Y^i}{\partial q^j} + \Gamma^i_{jr} X^j Y^r \right) \frac{\partial}{\partial q^i},$$

where $X = \sum_{i=1}^{n} X^i \partial/\partial q^i$ and $Y = \sum_{i=1}^{n} Y^i \partial/\partial q^i$.

Definition 2.2. *A **forced affine connection control system (FACCS)** is a control mechanical system given by $\Sigma = (Q, \nabla, Y, \mathscr{Y}, U)$ where*

- *Q is a smooth n–dimensional manifold called the configuration manifold,*
- *Y is a smooth time-dependent vector field along the projection $\tau_Q \colon TQ \to Q$, affine with respect to the velocities,*
- *$\mathscr{Y}$ is a set of k control vector fields on Q, and*
- *U is a measurable subset of $\mathbb{R}^k$.*

*A trajectory $\gamma \colon I \subset \mathbb{R} \to Q$ is **admissible for** Σ if $\dot{\gamma} \colon I \to TQ$ is absolutely continuous and there exists a measurable and bounded control $u \colon I \to U$ such that the dynamical equations of the control system Σ*

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = Y(t, \dot{\gamma}(t)) + \sum_{a=1}^{k} u_a(t) Y_a(\gamma(t)), \tag{2}$$

are fulfilled (for almost every $t \in I$).

The vector field Y includes all the non-controlled external forces; e.g., the potential and the non-potential forces. The assumption that Y is affine with respect to the velocities means that, in every local system of coordinates (q^i, v^i) on TQ, Y can be written as

$$Y(t, v_q) = Y_0(t, q) + \sum_{i=1}^{n} v^i Y^i(t, q).$$

Equation (2) can be rewritten as a first-order control-affine system on TQ,

$$\dot{\Upsilon}(t) = Z(\Upsilon(t)) + Y^V(t, \Upsilon(t)) + \sum_{a=1}^{k} u_a(t) Y_a^V(\Upsilon(t)), \tag{3}$$

where $\Upsilon \colon I \to TQ$ is such that $\tau_Q \circ \Upsilon = \gamma$, Z is the geodesic spray associated with the affine connection on Q and, for every $X \in \mathfrak{X}(Q)$, X^V denotes the vertical lift of X (see [10] for more details).

Apart from the usual Lie bracket that provides $\mathfrak{X}(Q)$ with a Lie algebra structure, one can associate with ∇ the following product in $\mathfrak{X}(Q)$.

Definition 2.3. *The **symmetric product** is the map*

$$\begin{aligned} \langle \cdot : \cdot \rangle \colon \mathfrak{X}(Q) \times \mathfrak{X}(Q) &\longrightarrow \mathfrak{X}(Q) \\ (X, Y) &\longmapsto \nabla_X Y + \nabla_Y X. \end{aligned}$$

It can be proved that

$$[Y_a^V, [Z, Y_b^V]] = \langle Y_a : Y_b \rangle^V \tag{4}$$

(see [1]).

3. Tracking problem

We consider here the problem arising when one tries to follow a particular trajectory on the configuration manifold, called *reference* or *target* trajectory, which is in general not a solution of the FACCS considered. A trajectory is successfully tracked if there exist solutions to the FACCS that approximate it arbitrarily well.

Consider any distance $\mathrm{d} \colon Q \times Q \to \mathbb{R}$ on Q whose corresponding metric topology coincides with the topology on Q.

Definition 3.1. *A curve $\gamma: I \to Q$ of class $\mathcal{C}^1$ is **trackable for the FACCS** Σ if, for every strictly positive tolerance ϵ, there exist a control $u^\epsilon \in L^\infty(I, U)$ and a solution $\xi^\epsilon: I \to Q$ to Σ corresponding to u^ϵ such that $\xi^\epsilon(0) = \gamma(0)$ and*

$$\mathrm{d}(\gamma(t), \xi^\epsilon(t)) < \epsilon$$

*for every $t \in I$. The trajectory is said to be **strongly trackable for** Σ if, in addition to the above requirements, for every $\epsilon > 0$ the approximating trajectory ξ^ϵ may be found also satisfying $\dot{\xi}^\epsilon(0) = \dot{\gamma}(0)$.*

*A control system Σ satisfies the **configuration tracking property (CTP)** (respectively, the **strong configuration tracking property (SCTP)**) if every curve on Q of class $\mathcal{C}^1$ is trackable (respectively, strongly trackable) for Σ.*

Remark 3.2. Since any $\mathcal{C}^1$ curve can be uniformly approximated, with arbitrary precision, by a smooth curve having the same tangent vector at its initial point, then Σ satisfies the CTP (respectively, the SCTP) if and only if every curve on Q of class $\mathcal{C}^\infty$ is trackable (respectively, strongly trackable) for Σ. $\diamond$

3.1. **Tracking results for control-linear systems.** A *control-linear system* (also called *driftless kinematic system*) on Q is a triple $(Q, \mathscr{X}, U)$ where $\mathscr{X}$ is a finite subset $\{X_1, \ldots, X_m\}$ of $\mathfrak{X}(Q)$ and U is a measurable subset of $\mathbb{R}^m$, identified with the control system

$$\dot{\gamma}(t) = \sum_{a=1}^{m} u_a(t) X_a(\gamma(t)), \qquad \gamma(t) \in Q,$$

where $u_1(\cdot), \ldots, u_m(\cdot)$ are measurable and bounded functions with $(u_1(t), \ldots, u_m(t)) \in U$ for every t.

Proposition 3.3 (See [11, 12]). *Let $X_1, \ldots, X_m$ be smooth vector fields on Q and take $\kappa \in \mathbb{N}$. Let $\{X_1, \ldots, X_{\hat{m}}\}$ be the set of all Lie brackets of the vector fields $X_1, \ldots, X_m$ of length less than or equal to κ. Assume that $\gamma : I \to Q$ is a $\mathcal{C}^\infty$ curve such that*

$$\dot{\gamma}(t) = \sum_{a=1}^{\hat{m}} w_a(t) X_a(\gamma(t)),$$

with $w : I \to \mathbb{R}^{\hat{m}}$ smooth. Then, for every $\epsilon > 0$ there exists a solution γ_ϵ of the control-linear system $(Q, \{X_1, \ldots, X_m\}, \mathbb{R}^m)$ with smooth control $u_\epsilon : I \to \mathbb{R}^m$ and initial condition $\gamma_\epsilon(0) = \gamma(0)$ such that $d(\gamma(t), \gamma_\epsilon(t)) < \epsilon$ for every $t \in I$.

From the above proposition we deduce the following result. (Similar arguments can be found in [13].)

Corollary 3.4. *If the Lie algebra $\mathrm{Lie}(X_1, \ldots, X_m)$ generated by $X_1, \ldots, X_m$ has constant rank on Q, then for every smooth curve $\gamma : I \to Q$ and for every $\epsilon > 0$ there exists a solution γ_ϵ of the control-linear system $(Q, \{X_1, \ldots, X_m\}, \mathbb{R}^m)$ with smooth control $u_\epsilon : I \to \mathbb{R}^m$ and initial condition $\gamma_\epsilon(0) = \gamma(0)$ such that $d(\gamma(t), \gamma_\epsilon(t)) < \epsilon$ for every $t \in I$.*

Proof. The proof works by covering the compact set $\gamma(I)$ by finitely many open sets $\Omega_1, \ldots, \Omega_K$ of Q such that for every $j = 1, \ldots, K$ there exists on Ω_j a basis of the distribution $\mathrm{Lie}(X_1, \ldots, X_m)$ made of Lie brackets of $X_1, \ldots, X_m$. Let κ be the maximum of the length of the brackets used to construct such bases and let $\{X_1, \ldots, X_{\hat{m}}\}$ be the set of all Lie brackets of the vector fields $X_1, \ldots, X_m$ of length less than or equal to κ. Then

$$\dot{\gamma}(t) = \sum_{a=1}^{\hat{m}} w_a(t) X_a(\gamma(t)),$$

with $w : I \to \mathbb{R}^{\hat{m}}$ smooth and we conclude by Proposition 3.3. ∎

3.2. **Previous strong configuration tracking results.** Conditions guaranteeing the SCTP have been obtained in [2], generalizing previous results presented in [1] (in particular Theorem 12.26) and in [3]. We recall them here below in a version adapted to what follows. The main difference of these statements from the ones of Theorem 4.4 and Corollary 4.7 in [2] is that here we focus on the strong configuration trackability of a given trajectory, instead of looking at the SCTP. The proof is however exactly the same, since the proof proposed in [2] is based on an argument where the target trajectory is fixed.

Proposition 3.5. *Let* $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ *be a FACCS. Construct the following set of vector fields on* Q:
$$
\begin{aligned}
\mathscr{K}_0 &= \overline{\mathrm{span}_{\mathcal{C}^\infty(Q)}\mathscr{Y}}, \\
\mathscr{K}_l &= \overline{\mathscr{K}_{l-1} - \mathrm{co}\left\{\langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{K}_{l-1})\right\}},
\end{aligned}
\tag{5}
$$
for $l \in \mathbb{N}$, *where, for* $A \subset \mathfrak{X}(Q)$, $\mathrm{L}(A) = A \cap (-A)$, $\mathrm{co}(A)$ *denotes the convex hull of* A, *and* $\overline{A}$ *is the closure of* A *in* $\mathfrak{X}(Q)$ *with respect to the topology of the uniform convergence on compact sets.*

Fix a smooth reference trajectory $\gamma_{\mathrm{ref}} : I \to Q$ *of class* $\mathcal{C}^\infty$. *Assume that there exist* $l, N \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_N \in C^\infty(I, [0, +\infty))$, *and* $Z_1, \ldots, Z_N \in \mathscr{K}_l$ *such that*
$$
\nabla_{\dot\gamma_{\mathrm{ref}}(t)}\dot\gamma_{\mathrm{ref}}(t) - Y(t, \dot\gamma_{\mathrm{ref}}(t)) = \sum_{a=1}^N \lambda_a(t) Z_a(\gamma_{\mathrm{ref}}(t)), \qquad \forall t \in I.
$$

Then γ_{ref} *is strongly trackable.*

Proposition 3.6. *Let* $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ *be a FACCS. Define the following sets of vector fields for* $l \in \mathbb{N}$,
$$
\begin{aligned}
\mathscr{Z}_0 &= \mathscr{Y}, \\
\mathscr{Z}_l &= \mathscr{Z}_{l-1} \cup \left\{\langle Z_a : Z_b \rangle \mid Z_a, Z_b \in \mathscr{Z}_{l-1}\right\}.
\end{aligned}
\tag{6}
$$
Assume that there exists $l \in \mathbb{N}$ *such that for each* $i \in \{0, \ldots, l-1\}$, *for each* $Z \in \mathscr{Z}_i$, $\langle Z : Z \rangle \in \mathrm{span}_{\mathcal{C}^\infty(Q)}\mathscr{Z}_i$.

Let $\mathscr{Z}_l = \{Z_1, \ldots, Z_N\}$. *Fix a smooth reference trajectory* $\gamma_{\mathrm{ref}} : I \to Q$ *of class* $\mathcal{C}^\infty$. *If there exist* $\lambda_1, \ldots, \lambda_N \in C^\infty(I, \mathbb{R})$ *such that*
$$
\nabla_{\dot\gamma_{\mathrm{ref}}(t)}\dot\gamma_{\mathrm{ref}}(t) - Y(t, \dot\gamma_{\mathrm{ref}}(t)) = \sum_{a=1}^N \lambda_a(t) Z_a(\gamma_{\mathrm{ref}}(t)), \qquad \forall t \in I,
$$

then γ_{ref} *is strongly trackable.*

4. A GENERALIZATION OF TRACKING CONDITIONS

Before introducing the new results about tracking, we need some technical lemmas described in Section 4.1 and to define kinematic reduction in Section 4.2. All that is necessary to prove the new theorem about trackability in Section 4.3.

4.1. **Pointwise and sectionwise characterisation of** $\mathscr{K}_l$. The results in this section, in the spirit of the classical Malgrange theorem (see [9]), aim at characterizing the sets $\mathscr{K}_l$ of sections of TQ, introduced above, in terms of iterated computations of subsets of TQ. Let us then associate with a family $\mathscr{Y} = \{Y_1, \ldots, Y_k\} \subset \mathfrak{X}(Q)$, in addition to the family $\mathscr{K}_l$, the family of subsets of TQ defined pointwise, for every $q \in Q$, as
$$
\widehat{\mathscr{K}_{0,q}} = \mathrm{span}_{\mathbb{R}}\mathscr{Y}(q),
\tag{7}
$$
$$
\widehat{\mathscr{K}_{l,q}} = \overline{\widehat{\mathscr{K}_{l-1,q}} - \mathrm{co}\left\{\langle Z : Z \rangle(q) \mid Z \in \mathfrak{X}(Q),\ Z(q') \in \mathrm{L}(\widehat{\mathscr{K}_{l-1,q'}}) \,\forall\, q' \in Q\right\}},
\tag{8}
$$
where for any $A \subset \mathfrak{X}(Q)$ we write $A(q) = \{Y(q) \mid Y \in A\} \subseteq T_q Q$.

Recall that each $\mathscr{K}_l$ is a convex cone in $\mathfrak{X}(Q)$ for the $\mathcal{C}^0_{\mathrm{loc}}$ topology (see [2, Proposition 4.1]). It is also clear that the recursive definition of $\widehat{\mathscr{K}_{l,q}}$ describes a closed convex cone of $T_q Q$.

We need a preliminary result to establish the equivalence between the two definitions.

Lemma 4.1. *Let $\mathscr{H}$ be a $\mathcal{C}^0_{\mathrm{loc}}$-closed set of $\mathfrak{X}(Q)$ and assume that $\mathscr{H}$ is closed with respect to finite linear combinations with coefficients in $\mathcal{C}^\infty(Q, [0, \infty))$. Then*

$$\mathscr{H} = \{ V \in \mathfrak{X}(Q) \mid V(q) \in \mathscr{H}(q) \quad \forall\, q \in Q \}. \tag{9}$$

Proof. The inclusion

$$\mathscr{H} \subseteq \{ V \in \mathfrak{X}(Q) \mid V(q) \in \mathscr{H}(q) \quad \forall\, q \in Q \}$$

is trivial and we are left to prove the opposite one.

If $V \in \mathfrak{X}(Q)$ and $V(q) \in \mathscr{H}(q)$ for all $q \in Q$, then for every $q \in Q$ there exists $W^q \in \mathscr{H}$ such that $W^q(q) = V(q)$. For all $\epsilon > 0$ there exists a closed neighbourhood $\Omega^{q,\epsilon}$ such that

$$\| W^q - V \|_{\infty, \Omega^{q,\epsilon}} \leq \epsilon,$$

where $\| \cdot \|_{\infty, \Omega^{q,\epsilon}}$ is the supremum norm restricted to $\Omega^{q,\epsilon}$, with respect to any fixed Riemannian structure on Q. For every $\epsilon > 0$, $\{\Omega^{q,\epsilon}\}_{q \in Q}$ is an open covering of Q. Let $(Q_n)_{n \in \mathbb{N}}$ be an increasing sequence of compact subsets whose interiors cover Q. For every $n \in \mathbb{N}$ there exists a finite covering $\Omega^{q_1, 1/n}, \ldots, \Omega^{q_{r_n}, 1/n}$ of Q_n and a partition of unity $a_1, \ldots, a_{r_n}$ subordinated to $\{\Omega^{q_i, 1/n}\}_{i=1}^{r_n}$ such that

$$\left\| \sum_{i=1}^{r_n} a_i W^{q_i} - V \right\|_{\infty, Q_n} \leq \frac{1}{n}.$$

In particular, the sequence $(\sum_{i=1}^{r_n} a_i W^{q_i})_{n \in \mathbb{N}}$ is contained in $\mathscr{H}$ and converges uniformly to V on compact sets. As $\mathscr{H}$ is closed, it follows that $V \in \mathscr{H}$. ∎

Proposition 4.2. *For every integer $l \geq 0$, let $\mathscr{K}_l$ and $\widehat{\mathscr{K}_{l,q}}$ be the convex cones defined in (5) and (8) respectively. Then*

$$\mathscr{K}_l = \{ V \in \mathfrak{X}(Q) \mid V(q) \in \mathscr{K}_l(q) \} \tag{10}$$

and

$$\mathscr{K}_l(q) = \widehat{\mathscr{K}_{l,q}}, \qquad \text{for every } q \in Q. \tag{11}$$

Proof. We first prove (10) by induction on l. According to Lemma 4.1, it is enough to prove that if $V \in \mathscr{K}_l$ and $a \in \mathcal{C}^\infty(Q, [0, \infty))$, then $aV \in \mathscr{K}_l$. The step $l = 0$ is trivial. Let $l \geq 1$ and assume that the property is true for $l - 1$. Since $\mathscr{K}_{l-1} - \mathrm{co}\,\{\langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{K}_{l-1})\}$ is $\mathcal{C}^0_{\mathrm{loc}}$-dense in $\mathscr{K}_l$ and $\{a^2 \mid a \in \mathcal{C}^\infty(Q, \mathbb{R})\}$ is $\mathcal{C}^0_{\mathrm{loc}}$-dense in $\mathcal{C}^\infty(Q, [0, \infty))$, it is enough to prove that $a^2 V \in \mathscr{K}_l$ for every $V \in \mathscr{K}_{l-1} - \mathrm{co}\,\{\langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{K}_{l-1})\}$ and $a \in \mathcal{C}^\infty(Q, \mathbb{R})$. Write $V = W - \sum_{j=1}^J \lambda_j \langle Z_j : Z_j \rangle$ with $W \in \mathscr{K}_{l-1}$, $\lambda_1, \ldots, \lambda_J > 0$ with $\sum_{j=1}^J \lambda_j = 1$ and $Z_1, \ldots, Z_J \in \mathrm{L}(\mathscr{K}_{l-1})$. By induction hypothesis $a^2 W \in \mathscr{K}_{l-1}$. Moreover, $\mathrm{L}(\mathscr{K}_{l-1})$ is also a $\mathcal{C}^0_{\mathrm{loc}}$-closed set of $\mathfrak{X}(Q)$, closed with respect to finite linear combinations with coefficients in $\mathcal{C}^\infty(Q, [0, \infty))$. Hence, applying Lemma 4.1 to $\mathscr{H} = \mathrm{L}(\mathscr{K}_{l-1})$ we deduce that $aZ_1, \ldots, aZ_J$ belong to $\mathrm{L}(\mathscr{K}_{l-1})$. It can be easily proved using (4) that

$$\langle aZ_j : aZ_j \rangle = a^2 \langle Z_j : Z_j \rangle + b_j Z_j$$

for some smooth function b_j. By induction hypothesis, $b_j Z_j$ is in $\mathscr{K}_{l-1}$. Hence, $-a^2 \langle Z_j : Z_j \rangle$ lies in $\mathscr{K}_{l-1} - \mathrm{co}\,\{\langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{K}_{l-1})\}$, concluding the proof of the identity $\mathscr{K}_l = \{ V \in \mathfrak{X}(Q) \mid V(q) \in \mathscr{K}_l(q) \}$.

As a consequence, if $Z(q) \in \mathrm{L}(\mathscr{K}_l(q))$ for all $q \in Q$, then $Z \in L(\mathscr{K}_l)$, which implies that

$$\mathrm{L}(\mathscr{K}_l) = \{Z \in \mathfrak{X}(Q) \mid Z(q) \in \mathrm{L}(\mathscr{K}_l(q)) \quad \forall \, q \in Q\}. \tag{12}$$

Let us now prove, again by induction on l, that (11) is true. The case $l = 0$ is trivial. Let us assume that (11) holds for $l - 1$, and let us prove it for l. According to (12) and the induction hypothesis,

$$\mathrm{L}(\mathscr{K}_{l-1}) = \{Z \in \mathfrak{X}(Q) \mid Z(q) \in \mathrm{L}(\widehat{\mathscr{K}}_{l-1,q}) \quad \forall \, q \in Q\}.$$

The definition of $\widehat{\mathscr{K}}_{l,q}$ then gives

$$\widehat{\mathscr{K}}_{l,q} = \overline{\mathscr{K}_{l-1}(q) - \mathrm{co}\left\{\langle Z \colon Z\rangle(q) \mid Z \in \mathrm{L}(\mathscr{K}_{l-1})\right\}}$$

which gives the result when compared with (5). ∎

4.2. **Kinematic reduction.** It is already known in the literature that to perform certain motion planning tasks it is useful to reduce a mechanical control system to a control-linear system in such a way that there exist relationships between the trajectories of both control systems. Before proceeding, we introduce some necessary definitions.

Let $\mathscr{X} = \{X_1, \ldots, X_m\} \subset \mathfrak{X}(Q)$ and consider the control-linear system $(Q, \mathscr{X}, \mathbb{R}^m)$ (defined in Section 3.1). Let us introduce the notations

$$\begin{aligned}
\mathrm{Sym}^{(0)}(\mathscr{Y})_q &= \mathrm{span}_{\mathbb{R}} \mathscr{Y}(q), \\
\mathrm{Sym}^{(1)}(\mathscr{Y})_q &= \mathrm{Sym}^{(0)}(\mathscr{Y})_q + \mathrm{span}_{\mathbb{R}}\{\langle W \colon Z\rangle(q) \mid W, Z \in \mathscr{Y}\}.
\end{aligned}$$

Definition 4.3. *Let $\Sigma = (Q, \nabla, 0, \mathscr{Y}, \mathbb{R}^k)$ be a FACCS. A driftless kinematic system $\Sigma_{\mathrm{kin}} = (Q, \mathscr{X}, \mathbb{R}^m)$ is a **kinematic reduction of** Σ if for every controlled trajectory $(\gamma, u_{\mathrm{kin}})$ of Σ_{kin} with u_{kin} smooth there exists u smooth such that (γ, u) is a controlled trajectory for Σ.*

Let us recall the following result from [1].

Theorem 4.4 ([1, Theorem 8.18]). *Let Σ and Σ_{kin} be as in Definition 4.3. Assume that $\mathscr{X}$ and $\mathscr{Y}$ generate constant-rank distributions. Then Σ_{kin} is a kinematic reduction of Σ if and only if $\mathrm{Sym}^{(1)} \mathscr{X}_q \subset \mathrm{span}_{\mathbb{R}} \mathscr{Y}(q)$ for every $q \in Q$.*

4.3. **A new criterion for trackability.** Let us now generalize the sufficient conditions for tracking given in Proposition 3.6.

Theorem 4.5. *Let $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ be a FACCS. Define the families $\mathscr{Z}_i$, $i \in \mathbb{N}$, of vector fields on Q as in (6). Assume that there exists $l \in \mathbb{N}$ such that*

(1) *$Y(t, p) \in \mathrm{span}_{\mathbb{R}} \mathscr{Z}_l(\tau_Q(p))$ for every $p \in TQ$;*
(2) *the distributions $\mathrm{span}_{\mathbb{R}} \mathscr{Z}_{l-1}$, $\mathrm{span}_{\mathbb{R}} \mathscr{Z}_l$, and $\mathrm{Lie}\,(\mathscr{Z}_{l-1})$ have constant rank;*
(3) *for all $i \in \{0, \ldots, l-1\}$ and $Z \in \mathscr{Z}_i$, $\langle Z \colon Z\rangle \in \mathrm{span}_{\mathbb{C}^\infty(Q)} \mathscr{Z}_i$.*

Fix a smooth reference trajectory $\gamma_{\mathrm{ref}} \colon I \to Q$ of class $\mathbb{C}^\infty$. If $\dot{\gamma}_{\mathrm{ref}}(t) \in \mathrm{Lie}_{\gamma_{\mathrm{ref}}(t)}\,(\mathscr{Z}_{l-1})$ for every $t \in I$, then γ_{ref} is trackable. In particular, if $\mathrm{Lie}_q\,(\mathscr{Z}_{l-1}) = T_q Q$ for every $q \in Q$ then the CTP holds.

Proof. Let l be as in the statement of the theorem and consider the FACCS

$$\Sigma_l = (Q, \nabla, 0, \mathscr{Z}_l, \mathbb{R}^{m_l}),$$

where m_l is the cardinality of $\mathscr{Z}_l$.

As recalled in Theorem 4.4,

$$\Sigma_{l-1,\mathrm{kin}} = (Q, \mathscr{Z}_{l-1}, \mathbb{R}^{m_{l-1}})$$

is a kinematic reduction of Σ_l, where m_{l-1} is the cardinality of $\mathscr{Z}_{l-1}$, since $\mathrm{Sym}^{(1)}(\mathscr{Z}_{l-1})_q = \mathrm{span}_{\mathbb{R}} \mathscr{Z}_l(q)$ for every $q \in Q$. Hence, every controlled trajectory of $\Sigma_{l-1,\mathrm{kin}}$ is also a controlled trajectory of Σ_l.

Since γ_{ref} is tangent to the distribution $\mathrm{Lie}(\mathscr{Z}_{l-1})$, we deduce from Corollary 3.4 that γ_{ref} can be tracked with arbitrary precision by trajectories of $\Sigma_{l-1,\mathrm{kin}}$.

Hence, given a positive tolerance ϵ, there exists a controlled trajectory (γ_1, u_1) of $\Sigma_{l-1,\mathrm{kin}}$ (still defined on the time-interval I) with $\gamma_1(0) = \gamma_{\mathrm{ref}}(0)$, $u_1 : I \to \mathbb{R}^{m_{l-1}}$ smooth, and such that

$$\mathrm{d}(\gamma_{\mathrm{ref}}(t), \gamma_1(t)) < \epsilon/2$$

for every time $t \in I$.

Now, by kinematic reduction, there exists $u_2 : I \to \mathbb{R}^{m_l}$ smooth such that (γ_1, u_2) is a controlled trajectory of Σ_l. Since the distribution generated by $\mathscr{Z}_l$ has constant rank, we can represent $Y(t, \gamma_1(t))$ as a linear combination of $Z_1(\gamma_1(t)), \ldots, Z_{m_l}(\gamma_1(t))$ with coefficients depending smoothly on the time. We recover that

$$\nabla_{\dot{\gamma}_1(t)}\dot{\gamma}_1(t) - Y(t, \gamma_1(t)) = \sum_{a=1}^{m_l} \lambda_a(t) Z_a(\gamma_1(t)), \qquad \forall t \in I,$$

with $\lambda_1, \ldots, \lambda_{m_l} \in C^\infty(I, \mathbb{R})$.

Applying now Proposition 3.6, we have that γ_1 is strongly trackable for Σ, and in particular there exists $u_3 : I \to \mathbb{R}^k$ such that the trajectory γ_3 of Σ corresponding to u_3 and with initial condition $\dot{\gamma}_3(0) = \dot{\gamma}_1(0)$ satisfies

$$\mathrm{d}(\gamma_1(t), \gamma_3(t)) < \epsilon/2$$

for every $t \in I$. We then conclude that γ_{ref} is trackable for Σ. ∎

In order to generalize the argument to situations in which the hypothesis that $\langle Z : Z \rangle \in \mathrm{span}_{C^\infty(Q)} \mathscr{Z}_i$ for every $Z \in \mathscr{Z}_i$ cannot be assumed, we introduce in the theorem below a new requirement on the linearity of the cones $\mathscr{K}_i$.

Theorem 4.6. *Let $\Sigma = (Q, \nabla, Y, \mathscr{V})$ be a FACCS. Define the families $\mathscr{K}_i$, $i \in \mathbb{N}$, of vector fields on Q as in (5). Assume that there exists $l \in \mathbb{N}$ such that*

- *$Y(t, p) \in \mathscr{K}_l(\tau_Q(p))$ for every $p \in TQ$;*
- *for all $q \in Q$, $\mathrm{L}(\mathscr{K}_{l-1}(q)) = \mathscr{K}_{l-1}(q)$ and $\mathrm{L}(\mathscr{K}_l(q)) = \mathscr{K}_l(q)$;*
- *the distributions $\mathscr{K}_{l-1}$ and $\mathscr{K}_l$ and $\mathrm{Lie}(\mathscr{K}_{l-1})$ have constant rank.*

Fix a smooth reference trajectory $\gamma_{\mathrm{ref}} : I \to Q$. If $\dot{\gamma}_{\mathrm{ref}}(t) \in \mathrm{Lie}_{\gamma_{\mathrm{ref}}(t)}(\mathscr{K}_{l-1})$ for every $t \in I$, then γ_{ref} is trackable.

In particular, if $\mathrm{Lie}_q(\mathscr{K}_{l-1}) = T_q Q$ for every $q \in Q$ then the CTP holds.

Proof. The reasoning works similarly to the one used in the proof of Theorem 4.5. The first step is then to check that $\mathrm{Sym}^{(1)}(\mathscr{K}_{l-1}) \subseteq \mathscr{K}_l$, allowing kinematic reduction arguments. By definition of $\mathscr{K}_l$ (see (5)), we know that $-\langle Z : Z \rangle$ lies in $\mathscr{K}_l$ for each $Z \in \mathrm{L}(\mathscr{K}_{l-1})$. Moreover, we deduce from (10) in Proposition 4.2 and the hypothesis $\mathrm{L}(\mathscr{K}_j(q)) = \mathscr{K}_j(q)$ for q in Q and $j = l-1, l$, that $\mathrm{L}(\mathscr{K}_j) = \mathscr{K}_j$ for $j = l-1, l$. Hence, $\pm\langle Z : Z \rangle$ lies in $\mathscr{K}_l$ for every Z in $\mathscr{K}_{l-1}$. As the symmetric product of any vector field can be written as a linear combination of symmetric products of vector fields with themselves,

$$\langle Z : W \rangle = \frac{1}{2}\left(\langle Z + W : Z + W \rangle - \langle Z : Z \rangle - \langle W : W \rangle\right),$$

we conclude that $\mathrm{Sym}^{(1)}(\mathscr{K}_{l-1}) \subseteq \mathscr{K}_l$.

Let $\mathscr{V} = \{V_1, \ldots, V_m\}$ be a set of generators of the distribution $q \mapsto \mathscr{K}_{l-1}(q)$ along γ_{ref}, i.e., $\mathscr{K}_{l-1}(\gamma_{\mathrm{ref}}(t)) = \mathrm{span}\{V_1(\gamma_{\mathrm{ref}}(t)), \ldots, V_m(\gamma_{\mathrm{ref}}(t))\}$ for every $t \in I$. It follows from Corollary 3.4 that the trajectories of $\Sigma_{l-1,\mathrm{kin}} = (Q, \mathscr{V}, \mathbb{R}^m)$ can track γ_{ref} with arbitrary precision. Hence, given a positive tolerance ϵ, there exists a controlled trajectory (γ_1, u_1) of $\Sigma_{l-1,\mathrm{kin}}$ (still defined on the time-interval I) with $\gamma_1(0) = \gamma_{\mathrm{ref}}(0)$ such that u_1 is smooth and

$$\mathrm{d}(\gamma_{\mathrm{ref}}(t), \gamma_1(t)) < \epsilon/2$$

for every time $t \in I$.

Let $\mathscr{U} = \{U_1, \ldots, U_r\}$ be a set of generators of $\mathscr{K}_l$ along γ_1. Since

$$\mathrm{Sym}^{(1)}(\mathscr{V})_q \subseteq \mathrm{Sym}^{(1)}(\mathscr{K}_{l-1})_q \subseteq \mathscr{K}_l(q) = \mathrm{span}_{\mathbb{R}}\{U_1(q), \ldots, U_r(q)\}$$

in a neighbourhood of the curve γ_{ref}, we deduce from Theorem 4.4 that there exists $u_2 : I \to \mathbb{R}^r$ such that (γ_1, u_2) is a controlled trajectory of $\Sigma_l = (Q, \nabla, 0, \mathscr{U}, \mathbb{R}^r)$.

Hence

$$\nabla_{\dot{\gamma}_1(t)}\dot{\gamma}_1(t) = \sum_{a=1}^{r} \eta_a^+(t) U_a(\gamma_1(t)) + \sum_{a=1}^{r} \eta_a^-(t)(-U_a(\gamma_1(t))), \qquad \forall\, t \in I,$$

with $\eta_1^+, \ldots, \eta_r^+, \eta_1^-, \ldots, \eta_r^- \in \mathcal{C}^\infty(I, [0, +\infty))$.

Since $\mathscr{K}_l$ has constant rank, we can represent $Y(t, \dot{\gamma}_1(t))$ as a linear combination of $U_1(\gamma_1(t))$, $\ldots, U_r(\gamma_1(t))$ with coefficients depending smoothly on the time. We recover that

$$\nabla_{\dot{\gamma}_1(t)}\dot{\gamma}_1(t) - Y(t, \dot{\gamma}_1(t)) = \sum_{a=1}^{r} \lambda_a^+(t) U_a(\gamma_1(t)) + \sum_{a=1}^{r} \lambda_a^-(t)(-U_a(\gamma_1(t))), \qquad \forall\, t \in I,$$

with $\lambda_1^+, \ldots, \lambda_r^+, \lambda_1^-, \ldots, \lambda_r^- \in C^\infty(I, [0, +\infty))$.

Applying now Proposition 3.5, we have that γ_1 is strongly trackable for Σ, and in particular there exists $u_3 : I \to \mathbb{R}^k$ such that the trajectory γ_3 of Σ corresponding to u_3 and with initial condition $\dot{\gamma}_3(0) = \dot{\gamma}_1(0)$ satisfies

$$\mathrm{d}(\gamma_1(t), \gamma_3(t)) < \epsilon/2$$

for every $t \in I$, and we conclude that γ_{ref} is trackable for Σ. ∎

4.4. **Example.** In this section we apply Theorem 4.5 to a control system studied in [2] and [3], completing the discussion on its trackability by tackling a case which was not covered by previously known criteria.

The system models a neutrally buoyant ellipsoidal vehicle immersed in a infinite volume fluid that is inviscid, incompressible and whose motion is irrotational. The dynamics are obtained through Kirchhoff equations [14] and have a particularly simple form due to some symmetry assumption on the distribution of mass (see [3] for details and also [15] for general overview of control motion in a potential fluid).

Consider the coordinates (ω, v) for the angular and linear velocity of the ellipsoid with respect to a body-fixed coordinate frame. Then the impulse (Π, P) of the system is given by

$$\begin{pmatrix} \Pi \\ P \end{pmatrix} = \mathcal{M} \begin{pmatrix} \omega \\ v \end{pmatrix}$$

where, under the symmetry assumptions mentioned above,

$$\mathcal{M} = \mathrm{diag}(J_1, J_2, J_3, M_1, M_2, M_3),$$

$\mathrm{diag}(J_1, J_2, J_3)$ is the usual inertia matrix, and M_1, M_2, M_3 take into account the mass of the submarine and the added masses due to the action of the fluid.

The configuration manifold Q for this problem is the Special Euclidean group or the group of rigid motions SE(3), which is homeomorphic to $SO(3) \times \mathbb{R}^3$. Let $(A, r) \in \mathrm{SE}(3)$ be the attitude and the position of the ellipsoid. Denote by $S : \mathbb{R}^3 \to \mathfrak{so}(3)$ the linear bijection between $\mathbb{R}^3$ and the linear algebra $\mathfrak{so}(3)$ of $SO(3)$ such that

$$S(x_1, x_2, x_3) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

The dynamics of the controlled system are given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = AS(\omega), \quad \frac{\mathrm{d}r}{\mathrm{d}t} = Av, \tag{13}$$

and

$$
\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \Pi \times \omega + P \times v + \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}, \quad \frac{\mathrm{d}P}{\mathrm{d}t} = P \times \omega + \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix}. \tag{14}
$$

The controls correspond to a linear acceleration along one of the three axes of the submarine and to two angular accelerations around the other two axes. It was proven in [3] that if $M_1 \neq M_2$ then system (13)–(14) satisfies the SCTP. In [2], moreover, based on general quantitative estimates of the convergence yielding the sufficient conditions for tracking recalled in Proposition 3.6, an explicit tracking algorithm was proposed.

The Lie group structure of the configuration manifold can be exploited to compute Lie brackets and symmetric products of left-invariant vector fields, and in particular of the control vector fields. (Otherwise, one can directly apply (4).) It turns out (see [1, 16] for details) that for $\eta_1 = \begin{pmatrix} S(w_1) & v_1 \\ 0 & 0 \end{pmatrix}$, $\eta_2 = \begin{pmatrix} S(w_2) & v_2 \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(3)$, identified with the corresponding left-invariant vector fields,

$$
[\eta_1, \eta_2] = \begin{pmatrix} [S(w_1), S(w_2)] & S(w_1)v_2 - S(w_2)v_1 \\ 0 & 0 \end{pmatrix}. \tag{15}
$$

Let $\{e_1, \dots, e_6\}$ be a basis adapted to the coordinates (A, r) so that $\eta_a = \eta_a^i e_i$ for $a = 1, 2$. It is then possible to compute the symmetric product as follows:

$$
\langle \eta_1 : \eta_2 \rangle = -\mathcal{M}^{-1}(\mathrm{ad}_{\eta_1}^* \mathcal{M}\eta_2 + \mathrm{ad}_{\eta_2}^* \mathcal{M}\eta_1), \tag{16}
$$

where $(\mathrm{ad}_\eta^* \alpha)\xi = \alpha(\mathrm{ad}_\eta \xi)$ for $\eta, \xi \in \mathfrak{se}(3)$, $\alpha \in \mathfrak{se}^*(3)$.

The structural constants with respect to the basis $\{e_1, \dots, e_6\}$ are defined as

$$
c_{ij}^k = [e_i, e_j]^k, \quad \gamma_{ij}^k = \langle e_i : e_j \rangle^k.
$$

By the expressions of the Lie bracket and the symmetric product given in (15) and (16), respectively, it follows that

$$
\gamma_{ij}^k = -\mathcal{M}^{hk}(\mathcal{M}_{il}c_{jh}^l + \mathcal{M}_{jl}c_{ih}^l),
$$

being $\mathcal{M}^{hk}$ the entries of the inverse matrix of $\mathcal{M}$.

One can easily compute that

$$
c_{23}^1 = c_{31}^2 = c_{12}^3 = c_{26}^4 = c_{53}^4 = c_{34}^5 = c_{61}^5 = c_{15}^6 = c_{42}^6 = 1,
$$
$$
c_{32}^1 = c_{13}^2 = c_{21}^3 = c_{62}^4 = c_{35}^4 = c_{43}^5 = c_{16}^5 = c_{51}^6 = c_{24}^6 = -1,
$$

and

$$
\gamma_{32}^1 = \gamma_{23}^1 = \frac{J_3 - J_2}{J_1}, \qquad \gamma_{56}^1 = \gamma_{65}^1 = \frac{M_3 - M_2}{J_1},
$$
$$
\gamma_{31}^2 = \gamma_{13}^2 = \frac{J_1 - J_3}{J_2}, \qquad \gamma_{46}^2 = \gamma_{64}^2 = \frac{M_1 - M_3}{J_2},
$$
$$
\gamma_{21}^3 = \gamma_{12}^3 = \frac{J_2 - J_1}{J_3}, \qquad \gamma_{45}^3 = \gamma_{54}^3 = \frac{M_2 - M_1}{J_3},
$$
$$
\gamma_{26}^4 = \gamma_{62}^4 = \frac{M_3}{M_1}, \qquad \gamma_{35}^4 = \gamma_{53}^4 = -\frac{M_2}{M_1},
$$
$$
\gamma_{16}^5 = \gamma_{61}^5 = -\frac{M_3}{M_2}, \qquad \gamma_{34}^5 = \gamma_{43}^5 = \frac{M_1}{M_2},
$$
$$
\gamma_{15}^6 = \gamma_{51}^6 = \frac{M_2}{M_3}, \qquad \gamma_{24}^6 = \gamma_{42}^6 = -\frac{M_1}{M_3},
$$

while all other structural constants are equal to zero.

Notice that the control vector fields are the vertical lift to $T\mathrm{SE}(3)$ of

$$
Y_1 = \frac{1}{J_1} e_1, \quad Y_2 = \frac{1}{J_2} e_2, \quad Y_3 = \frac{1}{M_3} e_6,
$$

where left-invariant vector fields are identified with elements of the Lie algebra of SE(3).

We focus here on the case that was left unanswered in [2] and [3], namely, the case where

$$J_1 = J_2, \qquad M_1 = M_2. \tag{17}$$

Such a case cannot be studied using the most general sufficient conditions for trackability given in [2] (recalled in Proposition 3.5), as illustrated by the computations here below.

Under condition (17), one easily computes that

$$\langle Y_1 : Y_2 \rangle = 0, \quad \langle Y_1 : Y_3 \rangle = -\frac{1}{J_1 M_1}\, e_5, \quad \langle Y_2 : Y_3 \rangle = \frac{1}{J_1 M_1}\, e_4.$$

Moreover,

$$\langle e_j : e_j \rangle = 0 \quad \text{ for } 1 \le j \le 6.$$

Let $Y_4 = -\dfrac{1}{J_1 M_1}\, e_5$ and $Y_5 = \dfrac{1}{J_1 M_1}\, e_4$. Hence, $\mathscr{Z}_1 = \{Y_j \mid 1 \le j \le 5\}$ and condition (3) in the statement of Theorem 4.5 is satisfied for $i = 0$.

Straightforward computations also give that $\langle Y_1 : Y_4 \rangle$ and $\langle Y_2 : Y_5 \rangle$ are proportional to Y_3, $\langle Y_3 : Y_4 \rangle$ is proportional to Y_1, $\langle Y_3 : Y_5 \rangle$ is proportional to Y_5, while

$$\langle Y_1 : Y_5 \rangle = \langle Y_2 : Y_4 \rangle = \langle Y_4 : Y_5 \rangle = 0.$$

Thus, $\mathscr{Z}_i(q) = \operatorname{span}_{\mathbb{R}}\{e_1, e_2, e_4, e_5, e_6\}_q$ for every $q \in \mathrm{SE}(3)$ and every $i \ge 1$, with condition (3) in the statement of Theorem 4.5 satisfied for every $i \ge 0$.

Note that

$$[Y_1, Y_2] = \frac{1}{J_1^2} e_3\,.$$

Thus, $\operatorname{Lie}_q(\mathscr{Z}_1) = T_q Q$ for every $q \in \mathrm{SE}(3)$. By Theorem 4.5 the configuration tracking property is guaranteed for these control vector fields. This completes the results in [2] and [3], allowing to conclude that system (13)–(14) satisfies the CTP for any choice of the (positive definite) diagonal inertial matrix $\mathcal{M}$.

Acknowledgements

References

[1] F. Bullo, A. D. Lewis, Geometric control of mechanical systems, Vol. 49 of Texts in Applied Mathematics, Springer-Verlag, New York, 2005, modeling, analysis, and design for simple mechanical control systems.

[2] M. Barbero-Liñán, M. Sigalotti, High-order sufficient conditions for configuration tracking of affine connection control systems, Systems Control Lett. 59 (8) (2010) 491–503.

[3] T. Chambrion, M. Sigalotti, Tracking control for an ellipsoidal submarine driven by Kirchhoff's laws, IEEE Trans. Automat. Control 53 (1) (2008) 339–349.

[4] M. Barbero-Liñán, M. C. Muñoz-Lecanda, Strict abnormal extremals in nonholonomic and kinematic control systems, Discrete Contin. Dyn. Syst. Ser. S 3 (1) (2010) 1–17. doi:10.3934/dcdss.2010.3.1.
URL http://dx.doi.org.proxy.queensu.ca/10.3934/dcdss.2010.3.1

[5] A. Bressan, Z. Wang, On the controllability of Lagrangian systems by active constraints, J. Differential Equations 247 (2) (2009) 543–563. doi:10.1016/j.jde.2009.01.014.
URL http://dx.doi.org/10.1016/j.jde.2009.01.014

[6] S. Martínez, J. Cortés, Motion control algorithms for simple mechanical systems with symmetry, Acta Appl. Math. 76 (3) (2003) 221–264. doi:10.1023/A:1023275502000.
URL http://dx.doi.org/10.1023/A:1023275502000

[7] A. A. Agrachev, A. V. Sarychev, Navier-Stokes equations: controllability by means of low modes forcing, J. Math. Fluid Mech. 7 (1) (2005) 108–152. doi:10.1007/s00021-004-0110-1.
URL http://dx.doi.org/10.1007/s00021-004-0110-1

[8] A. A. Agrachev, A. V. Sarychev, Controllability of 2D Euler and Navier-Stokes equations by degenerate forcing, Comm. Math. Phys. 265 (3) (2006) 673–697. doi:10.1007/s00220-006-0002-8.
URL http://dx.doi.org/10.1007/s00220-006-0002-8

[9] B. Malgrange, Ideals of differentiable functions, Tata Institute of Fundamental Research Studies in Mathematics, No. 3, Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1967.

[10] R. Abraham, J. E. Marsden, Foundations of mechanics, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978, second edition, revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman.

[11] W. Liu, An approximation algorithm for nonholonomic systems, SIAM J. Control Optim. 35 (4) (1997) 1328–1365. doi:10.1137/S0363012993260501.
URL `http://dx.doi.org/10.1137/S0363012993260501`

[12] H. J. Sussmann, W. Liu, Limits of highly oscillatory controls and approximation of general paths by admissible trajectories, in: Proceedings of the 30th IEEE Conference on Decision and Control, Brighton, UK, 1991.

[13] B. Jakubczyk, Introduction to geometric nonlinear control, controllability and lie bracket, in: Mathematical Control Theory, Vol. 8, ICTP Lecture Note Series, 2002.
URL `http://users.ictp.trieste.it/∼pub_off/lectures/vol8.html`

[14] H. Lamb, Hydrodynamics, sixth Edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993.

[15] A. Munnier, Locomotion of deformable bodies in an ideal fluid: Newtonian versus Lagrangian formalisms, J. Nonlinear Sci. 19 (6) (2009) 665–715. doi:10.1007/s00332-009-9047-0.
URL `http://dx.doi.org/10.1007/s00332-009-9047-0`

[16] F. Bullo, Invariant affine connections and controllability on Lie groups, technical Report for Geometric Mechanics, California Institute of Technology, http://www.cds.caltech.edu/ marsden/wiki/uploads/projects/geomech/Bullo1995.pdf (1995).